



TITLE:

Inequalities for Saigo's Fractional Calculus Operator (Study on Differential Operators and Integral Operators in Univalent Function Theory)

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Inequalities for Saigo's Fractional Calculus Operator

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Abstract

Let A be the class of functions $f(z)$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$ which are analytic in the open unit disk U . For $f(z) \in A$, the subclass $A(n, \delta)$ of A satisfying the coefficient inequalities $|a_k| \leq k^{n+\delta}$ ($k \geq 2$) is introduced. The object of the present paper is to derive some inequalities for Saigo's fractional calculus operator $I_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z) \in A(n, \delta)$.

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1. Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Saigo's fractional calculus operator $I_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z) \in A$ is defined in Srivastava, Saigo and Owa [7] (see also Saigo [5]) as follows.

Definition 1.1. For real numbers $\alpha > 0, \beta$ and η , the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z)$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} {}_2F_1 \left(\begin{matrix} \alpha+\beta, -\eta \\ \alpha \end{matrix} ; 1 - \frac{\zeta}{z} \right) f(\zeta) d\zeta, \quad (1.2)$$

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where $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin with the order

$$f(z) = O(|z|^\varepsilon) \quad (z \rightarrow 0),$$

where

$$\varepsilon > \max\{0, \beta - \eta\} - 1,$$

and the multiplicity of $(z - \zeta)^{\alpha-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Remark 1. It follows from Definition 1.1 that

$$I_{0,z}^{\alpha,-\alpha,\eta} f(z) = D_z^{-\alpha} f(z) = \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\alpha}} d\zeta, \quad (1.3)$$

when $\beta = -\alpha$, where $D_z^{-\alpha}$ is the fractional integral of order α defined by Owa [3].

Definition 1.2. For real numbers $0 \leq \alpha < 1, \beta$ and η , the fractional derivative operator $J_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z)$ is defined by

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{\alpha-\beta} \int_0^z (z - \zeta)^{-\alpha} {}_2F_1 \left(\begin{matrix} \beta - \alpha, 1 - \eta \\ 1 - \alpha \end{matrix}; 1 - \frac{\zeta}{z} \right) f(\zeta) d\zeta \right\} \quad (1.3)$$

and

$$J_{0,z}^{m+\alpha,\beta,\eta} f(z) = \frac{d^m}{dz^m} (J_{0,z}^{\alpha,\beta,\eta} f(z)) \quad (m = 0, 1, 2, \dots), \quad (1.5)$$

where $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin with the order

$$f(z) = O(|z|^\varepsilon) \quad (z \rightarrow 0),$$

where

$$\varepsilon > \max\{0, \beta - \eta\} - 1,$$

and the multiplicity of $(z - \zeta)^{-\alpha}$ is removed as in Definition 1.1 above.

Remark 2. We also note that, when $\beta = \alpha$,

$$J_{0,z}^{\alpha,\alpha,\eta} f(z) = D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ \int_0^z \frac{f(\zeta) d\zeta}{(z - \zeta)^\alpha} \right\} \quad (1.6)$$

and

$$J_{0,z}^{m+\alpha,\alpha,\eta} f(z) = D_z^{m+\alpha} f(z) = \frac{d^m}{dz^m} (D_z^\alpha f(z)), \quad (1.7)$$

where $D_z^\alpha f(z)$ and $D_z^{m+\alpha} f(z)$ are fractional derivatives of $f(z)$ defined by Owa [3].

A function $f(z) \in A$ is said to be starlike in U if it satisfies

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in U). \quad (1.8)$$

We denote by S^* the subclass of A consisting of all starlike functions in U . A function $f(z) \in A$ is said to be convex in U if it satisfies

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in U). \quad (1.9)$$

We also denote by K the subclass of A consisting of functions $f(z)$ which are convex in U . Note that $f(z) \in K$ if and only if $zf'(z) \in S^*$.

It is well-known that:

- (i) if $f(z) \in S^*$, then $|a_k| \leq k$ ($k = 2, 3, 4, \dots$), see e.g. [1],
and
(ii) if $f(z) \in K$, then $|a_k| \leq 1$ ($k = 2, 3, 4, \dots$), see e.g. [4].

Now, let $A(n, \delta)$ be the subclass of A consisting of all functions $f(z)$ which satisfy

$$|a_k| \leq k^{n+\delta} \quad (1.10)$$

for some $n = 0, 1, 2, \dots$, and for some $0 \leq \delta \leq 1$. Then we see that $S^* \subset A(1, 0)$ and $K \subset A(0, 0)$.

2. Inequalities

Let ${}_pF_q(z)$ be the generalized hypergeometric function defined by (for all details, see [2])

$${}_pF_q(z) \equiv {}_pF_q \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p \\ \beta_1, \beta_2, \dots, \beta_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \left(\frac{\prod_{j=1}^p (\alpha_j)_k}{\prod_{j=1}^q (\beta_j)_k} \right) \frac{z^k}{(1)_k}, \quad (2.1)$$

where $(\alpha_j)_k$ means the Pochhammer symbol defined by

$$(\alpha_j)_k = \begin{cases} 1 & (k = 0) \\ \alpha_j(\alpha_j + 1) \dots (\alpha_j + k - 1) & (k = 1, 2, 3, \dots) \end{cases} \quad (2.2)$$

In order to derive our inequalities for Saigo's fractional calculus operators, we need the following lemma due to Srivastava, Saigo and Owa [7].

Lemma 2.1. Let $\alpha > 0, \beta$ and η be real. Then, for $k > \max\{0, \beta - \eta\} - 1$,

$$I_{0,z}^{\alpha,\beta,\eta} z^\beta = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}. \quad (2.3)$$

We also have

Lemma 2.2. Let $0 \leq \alpha < 1, \beta$ and η be real. Then, for $k > \max\{0, \beta - \eta\} - 1$,

$$J_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k-\alpha+\eta+1)} z^{k-\beta} \quad (2.4)$$

$$J_{0,z}^{m+\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta-m+1)\Gamma(k-\alpha+\eta+1)} z^{k-\beta-m} \quad (m=0,1,2,\dots). \quad (2.5)$$

Our first inequality for $I_{0,z}^{\alpha,\beta,\eta} f(z)$ is contained in the following theorem.

Theorem 2.1. Let $\alpha > 0, \beta$ and η be real, and let $2 - \beta > 0, 2 - \beta + \eta > 0$, and $2 + \alpha + \eta > 0$. If $f(z) \in A(n, \delta)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} {}_{n+3}F_{n+2} \left(\begin{matrix} 2, \dots, 2, 2-\beta+\eta \\ 1, \dots, 1, 2-\beta, 2+\alpha+\eta \end{matrix} ; |z| \right) \quad (2.6)$$

for $0 < |z| < 1$.

P r o o f. Applying Lemma 2.1 for $f(z) \in A(n, \delta)$, we have

$$\begin{aligned} |I_{0,z}^{\alpha,\beta,\eta} f(z)| &= \left| \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} a_k z^{k-\beta} \right| \quad (a_1 = 1) \quad (2.7) \\ &\leq \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} k^{n+1} |z|^{k-\beta} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+2)\Gamma(k-\beta+\eta+2)}{\Gamma(k-\beta+2)\Gamma(k+\alpha+\eta+2)} (k+1)^{n+1} |z|^{k+1-\beta}. \end{aligned}$$

Since

$$\Gamma(k+\gamma) = \Gamma(\gamma)(\gamma)_k \quad (\gamma > 0) \quad \text{and} \quad k+1 = \frac{(2)_k}{(1)_k},$$

we obtain that

$$\begin{aligned} |I_{0,z}^{\alpha,\beta,\eta} f(z)| &\leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} \left(\sum_{k=0}^{\infty} \frac{((2)_k)^{n+2} (2-\beta+\eta)_k}{((1)_k)^n (2-\beta)_k (2+\alpha+\eta)_k} \frac{|z|^k}{(1)_k} \right) \quad (2.8) \\ &= \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} |z|^{1-\beta} {}_{n+3}F_{n+2} \left(\begin{matrix} 2, \dots, 2, 2-\beta+\eta \\ 1, \dots, 1, 2-\beta, 2+\alpha+\eta \end{matrix} ; |z| \right) \end{aligned}$$

which completes the proof of the theorem. ■

If we take $\beta = -\alpha$ in Theorem 2.1, then we have

Corollary 2.1. If $f(z) \in A(n, \delta)$, then

$$|D_z^{-\alpha} f(z)| \leq \frac{1}{\Gamma(2+\alpha)} |z|^{1+\alpha} {}_{n+2}F_{n+1} \left(\begin{matrix} 2, \dots, 2 \\ 1, \dots, 1, 2+\alpha \end{matrix} ; |z| \right) \quad (2.9)$$

for $0 < |z| < 1$ and $\alpha > 0$.

Taking special values of n and δ in Theorem 2.1, we derive the following corollary.

Corollary 2.2. Let $\alpha > 0, \beta$ and η be real, and let $2 - \beta > 0, 2 - \beta + \eta > 0$, and $2 + \alpha + \eta > 0$. If $f(z) \in A(1, 0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta)\Gamma(2 + \alpha + \eta)} |z|^{1-\beta} {}_4F_3 \left(\begin{matrix} 2, 2, 2, 2 - \beta + \eta \\ 1, 2 - \beta, 2 + \alpha + \eta \end{matrix} ; |z| \right) \quad (2.10)$$

for $0 < |z| < 1$. If $f(z) \in A(0, 0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta)\Gamma(2 + \alpha + \eta)} |z|^{1-\beta} {}_3F_2 \left(\begin{matrix} 2, 2, 2 - \beta + \eta \\ 2 - \beta, 2 + \alpha + \eta \end{matrix} ; |z| \right). \quad (2.11)$$

Similarly, for Saigo's fractional derivative operator $J_{0,z}^{\alpha,\beta,\eta} f(z)$ of $f(z)$, we have

Theorem 2.2. Let $0 \leq \alpha < 1, \beta$ and η be real, and let $2 - \beta - m > 0, 2 - \beta + \eta > 0, 2 - \alpha + \eta > 0$, and $m = 0, 1, 2, \dots$. If $f(z) \in A(n, \delta)$, then

$$\begin{aligned} |J_{0,z}^{m+\alpha,\beta,\eta} f(z)| &\leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta - m)\Gamma(2 - \alpha + \eta)} \\ &\times |z|^{1-\beta-m} {}_{n+3}F_{n+2} \left(\begin{matrix} 2, \dots, 2, 2 - \beta + \eta \\ 1, \dots, 1, 2 - \beta - m, 2 - \alpha + \eta \end{matrix} ; |z| \right) \end{aligned} \quad (2.12)$$

for $0 < |z| < 1$.

P r o o f. Since Lemma 2.2 implies that

$$J_{0,z}^{m+\alpha,\beta,\eta} f(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta-m+1)\Gamma(k-\alpha+\eta+1)} a_k z^{k-\beta-m}, \quad (2.13)$$

we easily see the inequality (2.12). ■

If we put $\beta = \alpha$ in Theorem 2.2, then we have

Corollary 2.3. If $f(z) \in A(n, \delta)$, then

$$|D_z^\alpha f(z)| \leq \frac{1}{\Gamma(2 - \alpha - m)} |z|^{1-\alpha-m} {}_{n+2}F_{n+1} \left(\begin{matrix} 2, \dots, 2 \\ 1, \dots, 1, 2 - \alpha - m \end{matrix} ; |z| \right) \quad (2.14)$$

for $0 < |z| < 1, 0 \leq \alpha < 1$ and $m = 0, 1$.

Taking special values for n and δ , we have

Corollary 2.4. Let $0 \leq \alpha < 1, \beta$ and η be real, and let $2 - \beta - m > 0, 2 - \beta + \eta > 0, 2 - \alpha + \eta > 0$ and $m = 0, 1, 2, \dots$. If $f(z) \in A(1, 0)$, then

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta - m)\Gamma(2 - \alpha + \eta)} |z|^{1-\beta-m} {}_4F_3 \left(\begin{matrix} 2, 2, 2, 2 - \beta + \eta \\ 1, 2 - \beta - m, 2 - \alpha + \eta \end{matrix} ; |z| \right) \quad (2.15)$$

for $0 < |z| < 1$. If $f(z) \in A(0, 0)$, then

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} |z|^{1-\beta-m} {}_3F_2 \left(\begin{matrix} 2, 2, 2-\beta+\eta \\ 2-\beta-m, 2-\alpha+\eta \end{matrix} ; |z| \right). \quad (2.16)$$

3. Appendix

We introduce the following formula for the generalized hypergeometric functions ([2]).

Lemma 3.1. (Srivastava [6]) *If m is a positive integer, then*

$${}_pF_q \left(\begin{matrix} \beta_1 + m, \alpha_2, \dots, \alpha_p \\ \beta_1, \beta_2, \dots, \beta_q \end{matrix} ; z \right) = \sum_{j=0}^m \binom{m}{j} \left(\frac{\prod_{s=2}^p (\alpha_s)_j}{\prod_{s=1}^q (\beta_s)_j} \right) z^j {}_{p-1}F_{q-1} \left(\begin{matrix} \alpha_2 + j, \dots, \alpha_p + j \\ \beta_2 + j, \dots, \beta_q + j \end{matrix} ; z \right) \quad (3.1)$$

In view of Lemma 3.1, for ${}_{n+3}F_{n+2}$ in Theorem 2.1, we see that

$$\begin{aligned} {}_{n+3}F_{n+2} \left(\begin{matrix} 2, \dots, 2, 2-\beta+\eta \\ 1, \dots, 1, 2-\beta, 2+\alpha+\eta \end{matrix} ; |z| \right) &= \sum_{j=0}^1 \binom{1}{j} \frac{((2)_j)^{n+1} (2-\beta+\eta)_j}{((1)_j)^n (2-\beta)_j (2+\alpha+\eta)_j} \\ &\quad \times |z|^j {}_{n+2}F_{n+1} \left(\begin{matrix} 2+j, \dots, 2+j, 2-\beta+\eta+j \\ 1+j, \dots, 1+j, 2-\beta+j, 2+\alpha+\eta+j \end{matrix} ; |z| \right) \\ &= {}_{n+2}F_{n+1} \left(\begin{matrix} 2, \dots, 2, 2-\beta+\eta \\ 1, \dots, 1, 2-\beta, 2+\alpha+\eta \end{matrix} ; |z| \right) \\ &\quad + \frac{2^{n+1} (2-\beta+\eta)}{(2-\beta)(2+\alpha+\eta)} |z| {}_{n+2}F_{n+1} \left(\begin{matrix} 3, \dots, 3, 3-\beta+\eta \\ 2, \dots, 2, 3-\beta, 3+\alpha+\eta \end{matrix} ; |z| \right) \\ &= {}_{n+1}F_n \left(\begin{matrix} 2, \dots, 2, 2-\beta+\eta \\ 1, \dots, 1, 2-\beta, 2+\alpha+\eta \end{matrix} ; |z| \right) \\ &\quad + \frac{3 \cdot 2^n (2-\beta+\eta)}{(2-\beta)(2+\alpha+\eta)} |z| {}_{n+1}F_n \left(\begin{matrix} 3, \dots, 3, 3-\beta+\eta \\ 2, \dots, 2, 3-\beta, 3+\alpha+\eta \end{matrix} ; |z| \right) \\ &\quad + \frac{4 \cdot 3^n (2-\beta+\eta)(3-\beta+\eta)}{(2-\beta)(3-\beta)(2+\alpha+\eta)(3+\alpha+\eta)} |z|^2 {}_{n+1}F_n \left(\begin{matrix} 4, \dots, 4, 4-\beta+\eta \\ 3, \dots, 3, 4-\beta, 4+\alpha+\eta \end{matrix} ; |z| \right) \\ &= \dots \end{aligned} \quad (3.2)$$

Therefore, Corollary 2.2 can be written as:

Corollary 3.1. Let $\alpha > 0, \beta$ and η be real, and let $2 - \beta > 0, 2 - \beta + \eta > 0$, and $2 + \alpha + \eta > 0$. If $f(z) \in A(1, 0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta)\Gamma(2 + \alpha + \eta)} {}_3F_2 \left(\begin{matrix} 2, 2, 2 - \beta + \eta \\ 2 - \beta, 2 + \alpha + \eta \end{matrix} ; |z| \right) + \frac{4\Gamma(2 - \beta + \eta)}{\Gamma(3 - \beta)\Gamma(3 + \alpha + \eta)} |z| {}_3F_2 \left(\begin{matrix} 3, 3, 3 - \beta + \eta \\ 3 - \beta, 3 + \alpha + \eta \end{matrix} ; |z| \right) \quad (3.3)$$

for $0 < |z| < 1$. Furthermore, let $\alpha > 0, \beta < 2$, and η be a positive integer. If $f(z) \in A(1, 0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta)\Gamma(2 + \alpha + \eta)} |z|^{1-\beta} \times \left\{ \sum_{j=0}^{\eta} \binom{\eta}{j} \left\{ \frac{((2)_j)^2}{(2 - \beta)_j(2 + \alpha + \eta)_j} {}_2F_1 \left(\begin{matrix} 2 + \eta, 2 + \eta \\ 2 + \alpha + 2\eta \end{matrix} ; |z| \right) + \frac{((2)_{j+1})^2(2 - \beta + \eta)}{(2 - \beta)_{j+1}(2 + \alpha + \eta)_{j+1}} |z| {}_2F_1 \left(\begin{matrix} 3 + \eta, 3 + \eta \\ 3 + \alpha + 2\eta \end{matrix} ; |z| \right) \right\} |z|^j \right\} \quad (3.4)$$

for $0 < |z| < 1$. If $f(z) \in A(0, 0)$, then

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta)\Gamma(2 + \alpha + \eta)} |z|^{1-\beta} \times \left\{ \sum_{j=0}^{\eta} \binom{\eta}{j} \frac{((2)_j)^2}{(2 + \alpha + \eta)_j} |z|^j {}_2F_1 \left(\begin{matrix} 2 + \eta, 2 + \eta \\ 2 + \alpha + 2\eta \end{matrix} ; |z| \right) \right\} \quad (3.5)$$

for $0 < |z| < 1$.

We also see from Corollary 2.4 that

Corollary 3.2. Let $0 \leq \alpha < 1, \beta$ and η be real, and let $2 - \beta - m > 0, 2 - \beta + \eta > 0$, $2 - \alpha + \eta > 0$ and $m = 0, 1, 2, \dots$. If $f(z) \in A(1, 0)$, then

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta - m)\Gamma(2 - \alpha + \eta)} |z|^{1-\beta-m} {}_3F_2 \left(\begin{matrix} 2, 2, 2 - \beta + \eta \\ 2 - \beta - m, 2 - \alpha + \eta \end{matrix} ; |z| \right) + \frac{4\Gamma(3 - \beta + \eta)}{\Gamma(3 - \beta - m)\Gamma(3 - \alpha + \eta)} |z|^{2-\beta-m} {}_3F_2 \left(\begin{matrix} 3, 3, 3 - \beta + \eta \\ 3 - \beta - m, 3 - \alpha + \eta \end{matrix} ; |z| \right) \quad (3.6)$$

for $0 < |z| < 1$. Furthermore, let $0 \leq \alpha < 1$ and $\beta < 2$ be real, and η be a positive integer, and let $2 - \beta - m > 0$ and $m = 0, 1, 2, \dots$. If $f(z) \in A(1, 0)$, then

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2 - \beta + \eta)}{\Gamma(2 - \beta - m)\Gamma(2 - \alpha + \eta)} |z|^{1-\beta-m}$$

$$\times \left\{ \sum_{j=0}^{\eta+m} \binom{\eta+m}{j} \left\{ \frac{((2)_j)^2}{(2-\beta-m)_j(2-\alpha+\eta)_j} {}_2F_1 \left(\begin{matrix} 2+\eta+m, 2+\eta+m \\ 2-\alpha+2\eta+m \end{matrix} ; |z| \right) \right. \right. \\ \left. \left. + \frac{((2)_{j+1})^2(2-\beta+\eta)}{(2-\beta-m)_{j+1}(2-\alpha+\eta)_{j+1}} |z| {}_2F_1 \left(\begin{matrix} 3+\eta+m, 3+\eta+m \\ 3-\alpha+2\eta+m \end{matrix} ; |z| \right) \right\} |z|^j \right\}$$

for $0 < |z| < 1$. If $f(z) \in A(0,0)$, then

$$|J_{0,z}^{m+\alpha,\beta,\eta} f(z)| \leq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta-m)\Gamma(2-\alpha+\eta)} |z|^{1-\beta-m} \quad (3.8)$$

$$\times \left\{ \sum_{j=0}^{\eta+m} \binom{\eta+m}{j} \frac{((2)_j)^2}{(2-\beta-m)_j(2-\alpha+\eta)_j} |z|^j {}_2F_1 \left(\begin{matrix} 2+\eta+m, 2+\eta+m \\ 2-\alpha+2\eta+m \end{matrix} ; |z| \right) \right\}$$

for $0 < |z| < 1$.

Further, we consider the case of $\beta = -\alpha$ in Corollary 2.2.

Corollary 3.3. Let $\alpha > 0$. If $f(z) \in A(1,0)$, then

$$|D_z^{-\alpha} f(z)| \leq \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)(1-|z|)^{3-\alpha}} \\ \times \left\{ (1-|z|) {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 2+\alpha \end{matrix} ; |z| \right) + \frac{4|z|}{2+\alpha} {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 3+\alpha \end{matrix} ; |z| \right) \right\} \quad (3.9)$$

for $z \in U$. If $f(z) \in A(0,0)$, then

$$|D_z^{-\alpha} f(z)| \leq \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 2+\alpha \end{matrix} ; |z| \right) \quad (3.10)$$

for $z \in U$.

P r o o f. Applying Lemma 3.1 and the formula (see [2])

$${}_2F_1 \left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; z \right) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1 \left(\begin{matrix} \gamma-\alpha, \gamma-\beta \\ \gamma \end{matrix} ; z \right) \quad (3.11)$$

we see that

$$|D_z^{-\alpha} f(z)| \leq \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} {}_3F_2 \left(\begin{matrix} 2, 2, 2 \\ 1, 2+\alpha \end{matrix} ; |z| \right) \\ = \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} \left\{ \sum_{j=0}^1 \binom{1}{j} \frac{(2)_j(2)_j}{(1)_j(2+\alpha)_j} |z|^j {}_2F_1 \left(\begin{matrix} 2+j, 2+j \\ 2+\alpha+j \end{matrix} ; |z| \right) \right\} \quad (3.12)$$

$$\begin{aligned}
&= \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)} \left\{ \sum_{j=0}^1 \binom{1}{j} \frac{(2)_j(2)_j}{(1)_j(2+\alpha)_j} \frac{|z|^j}{(1-|z|)^{2+j-\alpha}} {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 2+\alpha+j \end{matrix}; |z| \right) \right\} \\
&= \frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)(1-|z|^{3-\alpha})} \left\{ (1-|z|) {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 2+\alpha \end{matrix}; |z| \right) + \frac{4|z|}{2+\alpha} {}_2F_1 \left(\begin{matrix} \alpha, \alpha \\ 3+\alpha \end{matrix}; |z| \right) \right\}
\end{aligned}$$

for $z \in U$. ■

Letting $\beta = \alpha$ in Corollary 2.4, we have

Corollary 3.4. *let $0 \leq \alpha < 1$, $m = 0, 1$ and $\alpha + m < 2$. If $f(z) \in A(1, 0)$, then*

$$\begin{aligned}
|D_z^{m+\alpha} f(z)| &\leq \frac{|z|^{1-\alpha-m}}{\Gamma(2-\alpha-m)(1-|z|)^{3+\alpha+m}} \\
&\times \left\{ (1-|z|) {}_2F_1 \left(\begin{matrix} -\alpha-m, -\alpha-m \\ 2-\alpha-m \end{matrix}; |z| \right) + \frac{4|z|}{2-\alpha-m} {}_2F_1 \left(\begin{matrix} -\alpha-m, -\alpha-m \\ 3-\alpha-m \end{matrix}; |z| \right) \right\}
\end{aligned} \tag{3.13}$$

for $0 < |z| < 1$. If $f(z) \in A(0, 0)$, then

$$|D_z^{m+\alpha} f(z)| \leq \frac{|z|^{1-\alpha-m}}{\Gamma(2-\alpha-m)} {}_2F_1 \left(\begin{matrix} 2, 2 \\ 2-\alpha-m \end{matrix}; |z| \right) \tag{3.14}$$

for $0 < |z| < 1$.

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